

**PREDICTION OF THERMAL AND
MECHANICAL PROPERTIES
OF ECO-CONCRETES USING A BIPHASIC
HOMOGENIZATION APPROACH**

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Prediction of Thermal and Mechanical Properties of Eco-Concretes Using a Biphasic Homogenization Approach

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PREFACE

Because traditional building materials are bad for the environment, sustainable construction is now a big issue in civil engineering. In this context, environmental concrete, which uses recycled, waste-derived, or BioSource materials while still having good mechanical and thermal properties, has become a new and creative answer.

We need dependable theoretical tools that can tell us what these composite materials' effective properties will be in order to understand how they work. Homogenization theories help us model materials that are different by replacing their complicated microstructure with a simple, even medium, which allows us to predict the effective properties of composite materials and understand their behavior under various conditions.

This book talks about the theory behind biphasic homogenization methods for environmental concrete and how they are used in the real world. It wants to get more people to study sustainable building technologies and help engineers, researchers, and students better understand and predict how eco-friendly concrete materials will act when they are heated and cooled.

Keywords: Environmental concrete, sustainable construction, biphasic homogenization, composite materials, thermal behavior

29/03/2025

Dr. Abdelmoutalib BENFRID

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STATE OF THE ART REVIEW OF HOMOGENIZATION IN COMPOSITE MATERIALS

GENERAL INTRODUCTION

The building industry has used a lot of environmental concrete in the last several years and learnt a lot about it. The fundamental reason for this increased interest is the need to produce construction materials that are robust and long-lasting yet also good for the environment. This is why several studies have looked at how to make concrete better by adding new chemicals or making it stronger.

A number of studies are focused on improving concrete's ability to hold in heat and sound so that buildings are more pleasant and use less energy. Some individuals desire to make concrete buildings survive longer, operate better, and be stronger in all kinds of weather. Consequently, several research avenues integrating environmental sustainability, civil engineering, and materials science have developed in this domain.

Most people conceive of concrete as a mixture since it is made up of cement paste, aggregates, and other things. When fibers, recycled trash, metal particles, mineral powders, or bio-based elements are added to the composite structure, it becomes considerably more intricate. Because of this, it is quite hard for scientists to figure out how and why certain materials work.

Homogenization theories provide a valuable framework for analyzing heterogeneous composite materials, such as environmental concrete. These theories attempt to forecast the macroscopic characteristics of a composite material by analyzing the qualities of its individual components and their spatial distribution within the substance. With the correct mechanical and physical qualities, you may use homogenization techniques to make a complicated heterogeneous material seem like a simpler, more uniform one.

Homogenization models may be able to properly forecast how composite materials will act, but experiments are still required to make sure these predictions are correct and that they can be utilized in the real world. In this situation, homogenization models help scientists and engineers figure out how much of each new element to add to the concrete mix to acquire the optimum mechanical, thermal, or durability qualities.

Most techniques for homogenization are founded on the assumption that after the procedure, the heterogeneous composite material may be conceived of as homogenous and isotropic on a broad scale. This assumption substitutes the composite's complex microstructure with a comparable continuous medium with advantageous characteristics such as elastic modulus, thermal conductivity, and other mechanical or physical properties.

This book is composed of six chapters, each presenting a detailed study aimed at identifying and evaluating the thermal and mechanical properties of environmental concrete. The concept of a biphasic composite material is adopted throughout the chapters. In this approach,

conventional concrete is considered as the matrix phase, while the added materials whether mineral, metallic, recycled, or bio-based are treated as reinforcing phases within the composite structure.

Furthermore, each chapter discusses the theoretical background, limitations, and fundamental assumptions associated with the different homogenization models used in the analysis. Particular attention is given to the applicability of these models in predicting the effective properties of environmentally enhanced concrete materials, especially in terms of their strength, durability, and sustainability compared to traditional concrete materials.

The theoretical foundation of homogenization has been progressively developed through the contributions of numerous researchers. The models put up by Voigt and Reuss are among the first and most important. They provide upper and lower limits for the effective elastic characteristics of composite materials. Later advancements led to more advanced models, as those put out by Mooney, Eshelby, Hashin and Shtrikman, and Mori–Tanaka. These models made it easier to anticipate effective characteristics by taking into account how different phases interact and how the forms of inclusions affect them.

The models developed by researchers (Counto, Nielsen, Horio and Onogi, Piggot and Lavengood, Goettler, and Hirsch) are also very important in homogenization literatures but there are not examined in this book. Their analytical approaches have made it much easier for us to understand and predict how composite materials will behave mechanically.

There have been numerous research initiatives in this field that have led to this book. It is based on well-known scientific literature on composite materials and environmentally friendly concrete technology. This book tries to provide a complete scientific reference for researchers, engineers, and graduate students who wish to learn about and make concrete materials that are good for the environment. It achieves this by giving instances of theoretical models, how they are used in real life, and how they compare to each other.

The main purpose of this book is to explain and look at how biphasic homogenization methods can be used on concrete from the environment to guess its mechanical and thermal properties. The goal of this study is to use theoretical models, analytical formulations, and real-world examples to make it clearer how different types of reinforcing materials affect the overall performance of eco-concrete. The book also wants to show how to use homogenization to make things. By looking at the small-scale structure of composite materials, engineers and scientists may be able to learn more about their overall properties. This book has twelve chapters.. Each chapter has studies that engineers, researchers, and students may use as a scientific reference to make, develop, and improve sustainable concrete materials for modern buildings.

CHAPTER 1: VOIGT HOMOGENIZATION MODEL.

INTRODUCTION TO CHAPTER 1

The Voigt model for biphasic homogenization, which was first rule proposed in 1889, is one of the oldest and most important ways to homogenize no-homogenized composite materials in order to figure out their effective properties to more details we can consult paper (Voigt, 1889). This model assumes that the matrix and the reinforcing phase are perfectly bonded and are both under the same strain field. This means that when an outside force is put on the composite material, all of its parts are thought to deform in the same way. So, the phases of the composite are thought to be arranged in a parallel configuration. This means that the strain stays the same across all phases of the material, while the stresses are spread out based on how stiff each part is. Assuming this, the effective elastic modulus of the composite can be figured out by taking a weighted average of the elastic moduli of its different phases and their volume fractions.

The Voigt model is simple, but it has been used a lot to study composite materials because it gives a useful upper limit for the effective elastic properties of heterogeneous media. The model is especially useful when the reinforcing phase is evenly spread out and when the interaction between phases can be reasonably modeled as a uniform strain condition. In this situation of composite materials when environmental concrete is examine, the distribution of reinforcing particles or additives may not be perfectly uniform in general state , resulting in

discrepancies between rules predictions and experimental observations in laboratory.

A lot of scientists are working on improving the old Voigt model so that it can better explain why modern composite materials behave in strange ways. Researchers used a modified Voigt homogenization technique to study how concrete beams with small silica particles behave mechanically. The purpose of this study was to enhance the strength and durability of the composite material by incorporating fine silica particles into the concrete matrix. Researchers discovered that nanoparticles tended to stick together. This could change how well the concrete worked in general. These agglomeration processes alter the concept of a uniform distribution prevalent in traditional homogenization models. Researchers made a new version of the Voigt model that does a better job of showing how the mechanical properties of concrete with nano silica change when nanoparticles stick together (Harrat et al., 2021).

Subsequent studies examined the aggregation of silica particles utilized to enhance concrete slabs or plates structures. Researchers were particularly focused on the distribution of agglomerated particles in various regions of the slabs or plates, including the center and corners in plate, where structural pressures may vary significantly. The research demonstrated that the distribution and concentration of particle agglomerates significantly influence the stiffness and deformation of reinforced concrete slabs (Chatbi et al., 2022).

Researchers have utilized the Voigt homogenization model to look at how adding glass powder to concrete beams makes them bend. This research used glass powder as an eco-friendly ingredient to improve the durability of concrete mixes. It was also utilized to partly replace the usual cementitious materials in concrete. To achieve this, the Voigt model was utilized to figure out which mechanical characteristics of the composite material were useful. To form the composite material, the glass powder particles were combined with concrete. We looked at how reinforced concrete beams acted when they were put under mechanical stress in the study by Benfrid et al. (2024). We did this by utilizing the equivalent elastic modulus that we got via homogenization. (2024) by Benfrid et al.

A related study utilized the Voigt model to compare the anticipated elastic properties of concrete reinforced with glass powder to what really transpired in trials done under identical lab conditions. By comparing the two, researchers were able to figure out how dependable the homogenization process was for eco-friendly concrete materials. The results showed that the Voigt model may provide very good first guesses about the effective elastic properties. But things like the distribution of particles, the amount of space between them, and how the matrix and reinforcement phases interact might cause variations (Benfrid et al., 2024).

The title of this chapter, "Voigt Homogenization Model," comes from the hypothesis that Voigt came up with. It shows you how to use the Voigt homogenization method to learn about the thermal and

mechanical properties of concrete that is utilized in the environment. There are also some real-world instances and problems that have been spoken about to help make the idea easier to understand and the procedure clearer. These examples will help you understand and use the Voigt model better when you look at real-world materials that are better for the environment.

1. VOIGT RULE OF HOMOGENIZATION

1.1. Voigt Model to determinate mechanical propertie

We assume that the reinforcement and the matrix are touching each other in the medium and are lined up in a series to get the elastic modulus. This lets us use a representative volume element (RVE) to show the material.

$$E = \frac{E_r \times V_r}{V} + \frac{E_m \times (1 - V_r)}{V} \quad (1)$$

The total volume of the composite material is equal to the sum of the volume fractions of the reinforcement and the matrix, which is equal to 1.

$$V_r + V_m = 1 \quad (2)$$

The bulk modulus K can be expressed as a function of Young's modulus E and Poisson's ratio ν for an isotropic elastic material.

$$K = \frac{E}{3(1 - 2\nu)} \quad (3)$$

The shear modulus G characterizes the material's resistance to shear deformation.

$$G = \frac{E}{2(1 + \nu)} \quad (4)$$

The effective Poisson's ratio of the composite material is determined using a summation method that accounts for the respective contributions of the reinforcement and the matrix.

$$\nu = V_r \nu_r + (1 - V_r) \nu_m \quad (5)$$

1.2. Thermal Homogenization with Voigt Model

Thermal homogenization aims to estimate the effective thermal conductivity of a composite material based on the thermal conductivities and volume fractions of its constituent phases.

$$\lambda = \frac{\lambda_r \times V_r}{V} + \frac{\lambda_m \times (1 - V_r)}{V} \quad (6)$$

The effective coefficient of thermal expansion is determined by a summation approach that accounts for the respective contributions of the reinforcement and the matrix according to their volume fractions.

$$\alpha = V_r \alpha_r + (1 - V_r) \alpha_m \quad (7)$$

1.3. Physical Homogenization with Voigt Model

Physical homogenization aims to evaluate the effective physical properties of composite materials based on the properties and volume fractions of their constituent phases. The effective density of the composite material is determined as the weighted average of the densities of the reinforcement and the matrix according to their respective volume fractions.

$$\rho = V_r \rho_r + (1 - V_r) \rho_m \quad (8)$$

The total mass of the composite material is equal to the sum of the masses of the reinforcement and the matrix.

$$m = m_r + m_m \quad (9)$$

1.4. Exercises and Applications

Application 1:

Determine with Voigt model the effective mechanical properties by mechanical homogenization of an ordinary concrete matrix ($E_m=20$ GPa, $\nu_m=0.2$) containing steel inclusions as reinforcement ($E_r=210$ GPa, $\nu_r=0.3$).when ($V_r=0\%$ to 100%)

Table 1: The effective mechanical properties of eco concrete reinforced with steel inclusions.

V_r (%)	E (GPa)	ν	K (GPa)	G (GPa)
0	20.00	0.20	11.11	8.33
10	39.00	0.21	22.41	16.12
20	58.00	0.22	34.52	23.77
30	77.00	0.23	47.53	31.30
40	96.00	0.24	61.54	38.71
50	115.00	0.25	76.67	46.00
60	134.00	0.26	93.06	53.17
70	153.00	0.27	110.87	60.24
80	172.00	0.28	130.30	67.19
90	191.00	0.29	151.59	74.03
100	210.00	0.30	175.00	80.77

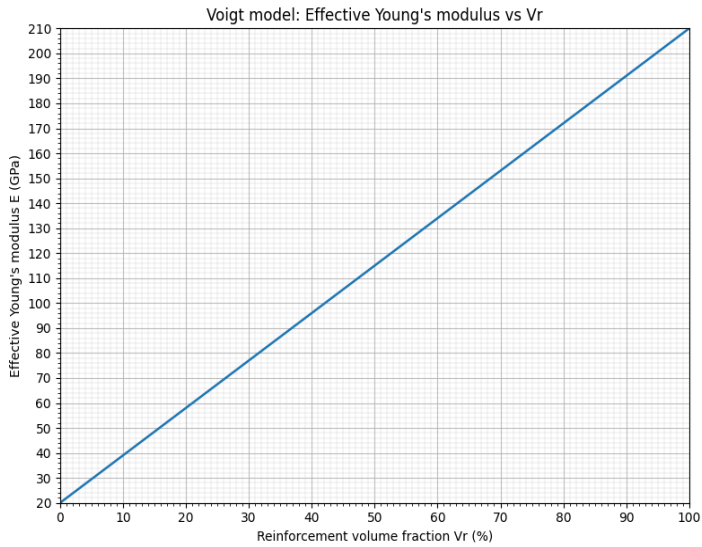


Figure 1: The effective Young's modulus of eco concrete reinforced with steel inclusions.

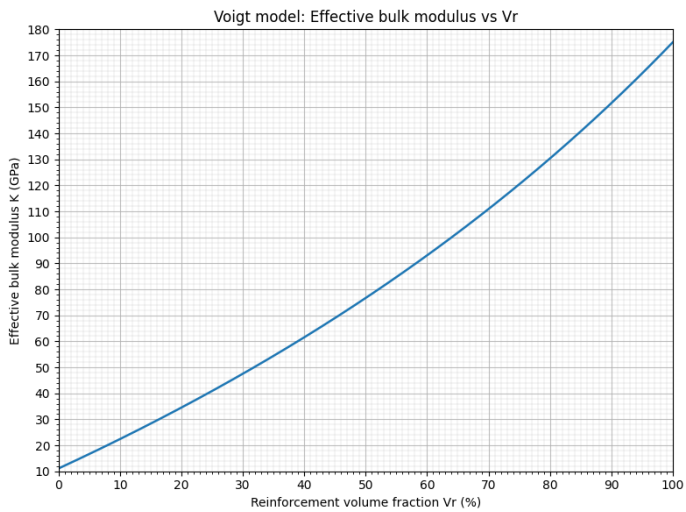


Figure 2: The effective bulk modulus of eco concrete reinforced with steel inclusions.

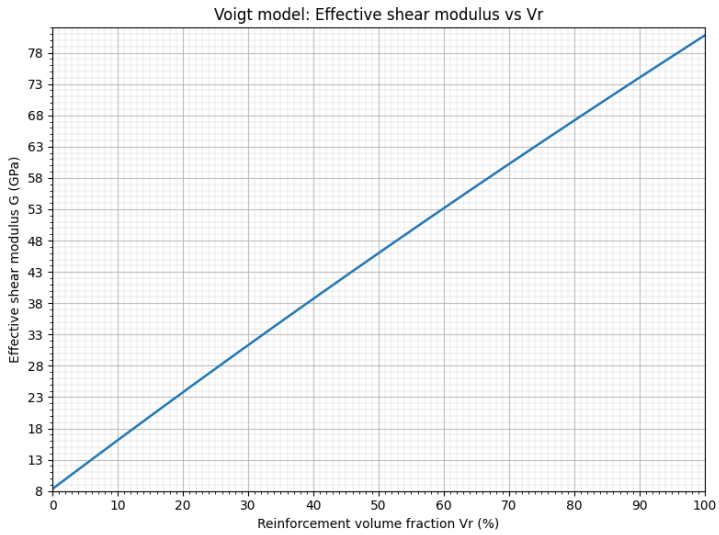


Figure 3: The effective shear modulus of eco concrete reinforced with steel inclusions

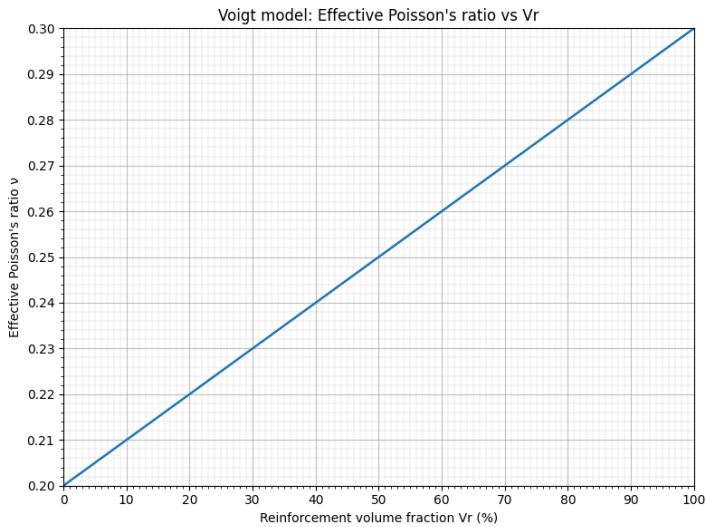


Figure 4: The effective Poisson's ratio of eco concrete reinforced with steel inclusions

Application 2:

Using the Voigt model, determine the effective thermal and physical properties of an ordinary concrete matrix ($\lambda_m=1.65 \text{ W/m}^*\text{C}$, $\alpha_m=10\times 10^{-6} \text{ }^\circ\text{C}^{-1}$) reinforced with wood waste ($\lambda_r=0.12 \text{ W/m}^*\text{C}$, $\alpha_r=5\times 10^{-6}\text{C}^{-1}$), for reinforcement volume fractions V_r varying from 0% to 100%.

Table 2: The effective thermal properties of eco concrete reinforced with wood inclusions.

V_r (%)	λ (W/m*°C)	$\alpha\times 10^{-6}$ (C⁻¹)
0	1.650	10.0
10	1.497	9.5
20	1.344	9.0
30	1.191	8.5
40	1.038	8.0
50	0.885	7.5
60	0.732	7.0
70	0.579	6.5
80	0.426	6.0
90	0.273	5.5
100	0.120	5.0

1.5. Limitations of the Voigt Model

The Voigt model presents several limitations when applied to non-homogenous composite materials. It is based on the assumption that all phases of the composite are subjected to the same strain, known as the uniform strain condition, which does not always represent the actual mechanical behavior of composite materials. In composite materials, the amount of strain that various phases between the matrix and reinforcements experience might vary according on how they are organized in space deviators and how they act mechanically. The model also implies that the matrix and reinforcement are set up in a perfect parallel method, which doesn't happen very frequently in actual materials.

Due to the fact that these simplifying assumptions are made, the Voigt model often generates an upper limit estimate of the effective elastic characteristics of the composite material. Due to the fact that it is linear, it could give the impression of being more rigid than it really is. In addition, the technique does not take into account significant aspects, such as the manner in which inclusions interact with the matrix that surrounds them or the manner in which the form, orientation, or spatial distribution of reinforcing particles affects the model. The Voigt model is simple to use and helpful for conducting research in a short amount of time; nevertheless, it does not always yield solid predictions for complex composite materials that have microstructures that are not the same across the whole material.

1.6. Conclusion of chapter 1

You can use the Voigt model, which is a simple and common way to mix materials, to find out the effective properties of composite materials. By looking at the intrinsic properties and volume fractions of the matrix and reinforcing phases, you can find out the material's mechanical, thermal, and physical properties. The model makes the analysis easier by assuming that all parts of the composite experience the same strain. This lets the effective properties be shown as a weighted average of the properties of each phase. On the other hand, the Voigt model can give you an estimate of the composite material's effective properties that is higher than the actual properties. This is mostly because it makes things easier by assuming that the strain is the same everywhere. Even with the problems mentioned above, the model is still a useful tool for getting a first look at and understanding how different materials behave, such as environmental concrete and other composite systems.

CHAPTER 2: REUSS HOMOGENIZATION MODEL

INTRODUCTION TO CHAPTER 2

The Reuss model, first out in 1929, is another simple way to homogenize that is often used to study composite materials that are not uniform, especially those that are not very stiff (Reuss, 1929). This concept posits that the matrix and reinforcement phases are organized in a series arrangement and experience an identical stress field, however the strain differs in each phase of the composite material. Based on this assumption, the effective attributes of the composite are based on the harmonic average of the qualities of each part.

It was interesting to see how The Reuss model's results matched Benbakhti et al. (2023)'s predictions about the use of eco-friendly building materials. Finding ways to lessen the harm that the construction industry causes to the environment is the study's main goal. One option is to recycle industrial waste, especially glass waste. Concrete mixes are increasingly using glass waste in place of some of the cement. It is likely that this trend will continue. The main goal of the empirical study was to determine how adding glass powder, which has an average particle size of 2.79 μm , affects the mechanical properties of eco-concrete. They experimented with replacement glass powder ranging from 5% to 35% in order to accomplish this. The results showed that the Reuss–Voigt homogenization method is an effective way to test the strength of eco-concrete. Benbakhti et al. (2023) report that their experimental results agreed with theoretical predictions (Benbakhti et al., 2023).

Benfrid et al. conducted an analytical study in 2024 to ascertain the effective modulus of elasticity of eco-concrete incorporating glass powder in a mixed state (Benfrid et al., 2024). This study compared real experimental results with different ways to improve new homogenization models, like the Voigt model and the Reuss–Voigt model. The findings indicated that the Reuss–Voigt model yielded the most accurate predictions, particularly for cement replacement ratios ranging from 5% to 20% and fine aggregate replacement ratios up to 25%, especially following 56 days of curing. These results show how important homogenization methods are for making and improving concrete materials that are good for the environment.

Benfrid et al. (2025) investigated the long-lasting enhancement of thermal and mechanical properties of eco-concrete through the incorporation of glass nano powder (GNP). This study used different models of analytical homogenization to see how well the material worked. We used the Maxwell–Eucken model to find out which thermal conductivity worked. We used the Luo model to find out how flexible concrete structures with nano-reinforcement were. The results showed that the material's thermal properties got a lot better when 30% glass nano powder was added. The thermal conductivity went up by almost 15%, the thermal resistance went up by about 25%, and the thermal expansion and thermal transmittance went down by about 30% and 25%, respectively. These findings indicate that eco-concrete containing nano powder is suitable for construction projects requiring reduced energy consumption.

In a later study in 2025, Benfrid used the Reuss principle to look into the thermo-mechanical properties of PVC reservoirs that were strengthened with short glass fibers (Benfrid, 2025). The study investigated the effects of thermal buckling instability on composite panels, specifically those containing 10% and 20% short glass fibers within PVC. They made a computer model to see how adding fiber reinforcement changes how easily the structure bends when it gets hot. The results showed that adding short glass fibers could make the composite structure less stable at some temperatures.

This chapter talks about the Reuss model and how it can make things stronger and cooler at the same time. The model shows that the matrix and reinforcement phases are all in the same stress field and are all in a row. The Reuss model shows us the least useful properties of composite materials that aren't the same all the way through. It's easier now, but it's still a good way to learn about how composite materials work and how different types of reinforcement affect how well concrete works in different situations.

2. REUSS HOMOGENIZATION

2.1. Reuss Model to evaluate the mechanical properties

In order to calculate the elastic modulus, it is necessary to make the assumption that the reinforcement and the matrix are in touch with one another inside the medium and that they are organized in a series configuration. This arrangement makes it possible to describe the composite material using a representative volume element (RVE).

$$E = \frac{1}{\frac{1 - V_r}{E_m} + \frac{V_r}{E_r}}$$

or

$$E = \frac{E_m E_r}{E_r(1 - V_r) + E_m V_r} \quad (10)$$

The total volume of the composite material is equal to the sum of the volume fractions of the reinforcement and the matrix, which is equal to 1, as noted in Equation (2) of this book. The bulk modulus K can be expressed as a function of Young's modulus E and Poisson's ratio ν for an isotropic elastic material, as presented in Equation (3). The shear modulus G characterizes the material's resistance to shear deformation, as shown in Equation (4). The effective Poisson's ratio of the composite material is determined using a summation method that accounts for the respective contributions of the reinforcement and the matrix, as indicated in Equation (5). For a detailed presentation of these formulas, it is preferable to refer to Chapter 1, where these relations are introduced and discussed in detail.

2.2. Thermal Homogenization with Reuss Model

Thermal homogenization aims to estimate the effective thermal conductivity of a composite material based on the thermal conductivities and volume fractions of its constituent phases.

$$\lambda = \frac{\lambda_r \times V_r}{V} + \frac{\lambda_m \times (1 - V_r)}{V} \quad (6)$$

The effective coefficient of thermal expansion is determined by a summation approach that accounts for the respective contributions of the reinforcement and the matrix according to their volume fractions.

$$\alpha = V_r \alpha_r + (1 - V_r) \alpha_m \quad (7)$$

2.3. Physical Homogenization with Reuss Model

Physical homogenization aims to determine the effective physical properties of composite materials by considering the properties and volume fractions of their constituent phases. The effective density of the composite is calculated as the weighted average of the densities of the reinforcement and the matrix according to their respective volume fractions, as shown in Equation (8). In addition, the total mass of the composite material is equal to the sum of the masses of the reinforcement and the matrix, as expressed in Equation (9).

2.4. Exercises and Applications

Application 3:

The Reuss homogenization model is used to figure out the effective mechanical properties of a concrete matrix with a volume of 0.2 and a capacity of 20 gigapascals. Glass powder, which has a strength of 70 gigapascals and a coefficient of 0.3, strengthens the matrix. The volume fractions of the reinforcement, which are called V_r , can be anywhere from 0% to 100%.

Table 3: The effective mechanical properties of eco concrete reinforced with glass powder.

V_r (%)	E (GPa)	ν	K (GPa)	G (GPa)
0	20.00	0.20	11.11	8.33
10	21.54	0.21	12.38	8.90
20	23.33	0.22	13.89	9.56
30	25.45	0.23	15.71	10.35
40	28.00	0.24	17.95	11.29
50	31.11	0.25	20.74	12.44
60	35.00	0.26	24.31	13.89
70	40.00	0.27	29.20	15.75
80	46.67	0.28	35.35	18.23
90	56.00	0.29	44.44	21.71
100	70.00	0.30	58.33	26.92

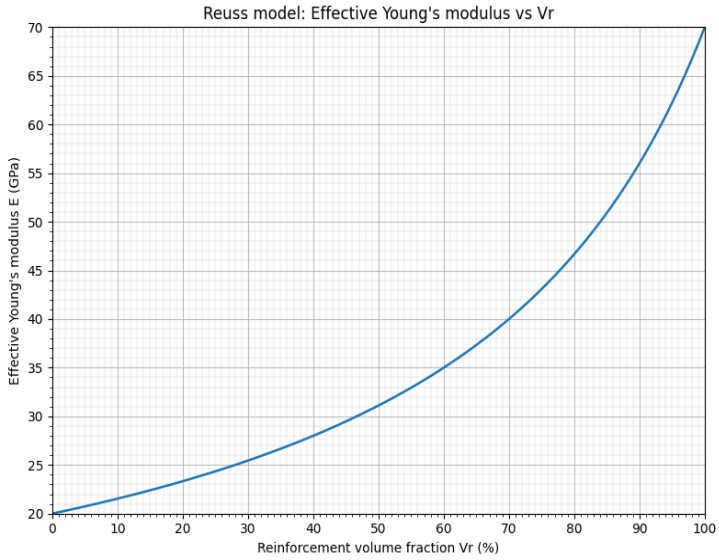


Figure 5: The effective Young's modulus of eco concrete reinforced with glass powder.

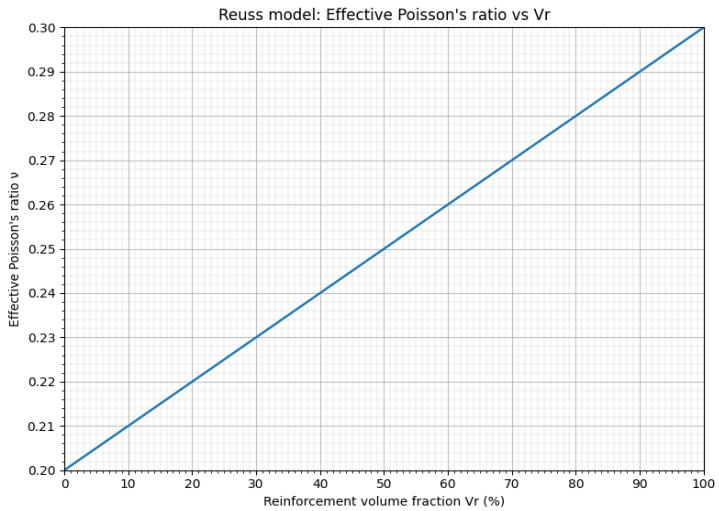


Figure 6: The effective bulk modulus of eco concrete reinforced with glass powder.

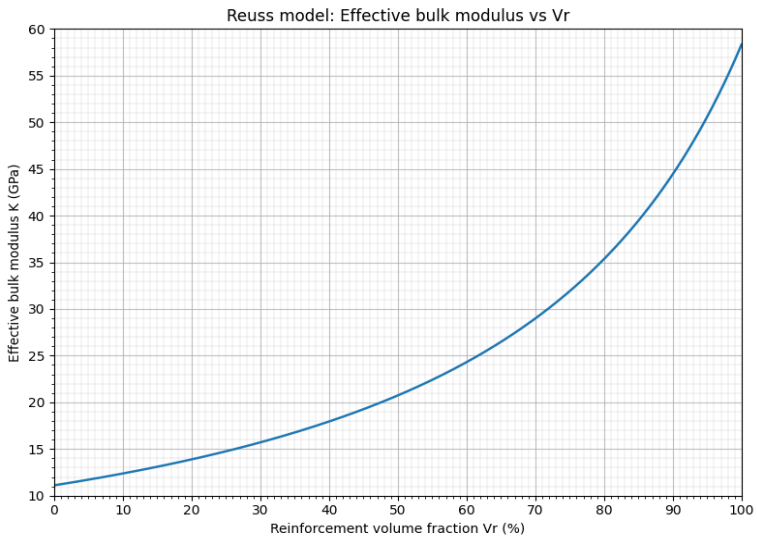


Figure 7: The effective shear modulus of eco concrete reinforced with glass powder.

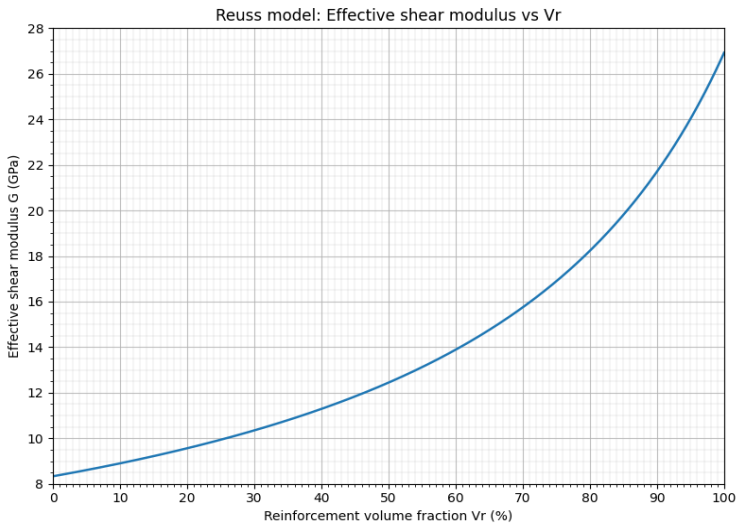


Figure 8: The effective Poisson's ratio of eco concrete reinforced with glass powder.

Application 4:

Apply the Reuss homogenization model to evaluate the effective thermal and physical properties of a concrete matrix ($\lambda_m=1.65$ W/m $^{\circ}$ C, $\alpha_m=10 \times 10^{-6}$ $^{\circ}$ C $^{-1}$) reinforced with carton (cardboard) waste ($\lambda_r=0.037$ W/m $^{\circ}$ C, $\alpha_r=9 \times 10^{-6}$ $^{\circ}$ C $^{-1}$), for reinforcement volume fractions V_r varying from 0% to 100%.

Table 4: The effective thermal properties of eco concrete reinforced with carton (cardboard) waste.

V_r (%)	λ (W/m$^{\circ}$C)	α×10$^{-6}$ (C$^{-1}$)
0	1.6500	10.0
10	0.2926	9.9
20	0.1577	9.8
30	0.1088	9.7
40	0.0835	9.6
50	0.0680	9.5
60	0.0575	9.4
70	0.0498	9.3
80	0.0440	9.2
90	0.0399	9.1
100	0.0370	9.0

2.5. Limitations of the Reuss Model

The Reuss model also presents several limitations when applied to heterogeneous composite materials. It assumes all phases of the composite are subjected to the same stress field, known as the uniform stress condition, which may misrepresent the real mechanical behavior of composite materials. In practice, the stress distribution within a heterogeneous material can vary significantly depending on the mechanical properties and spatial arrangement of its constituents.

Moreover, the model assumes that the matrix and reinforcement phases are arranged in a series configuration, an idealized condition that rarely reflects the complex microstructure of real composite materials. As a consequence of these simplifying assumptions, the Reuss model generally provides a lower bound estimate of the effective elastic properties of the composite, which may lead to an underestimation of its stiffness.

In addition, the model does not take into account several important factors, such as the shape, orientation, spatial distribution, and interaction of inclusions within the matrix. These parameters can have a significant influence on the global mechanical response of composite materials, potentially leading to inaccuracies in the predictions made by the Reuss model if they are not considered. Therefore, although the Reuss model is simple and easy to implement, its predictions often need to be complemented by other homogenization models, such as the Voigt model or the Hashin-Shtrikman bounds, to obtain more accurate and

realistic estimations of the effective properties of heterogeneous composites, particularly in applications where precise material behavior is critical.

2.6. Conclusion of chapter 2

The Voigt and Reuss models are two well-known ways to homogenize materials that are often used to figure out the effective properties of composite materials. The Voigt model assumes that the strain is evenly spread out across the composite. This usually leads to an upper bound estimate of the effective properties and may make the material seem stiffer than it really is. The Reuss model, on the other hand, assumes that stress is evenly spread out across all parts of the composite. This means that it gives a lower bound estimate of the effective properties, which could make the composite seem stiffer than it really is.

These two models create a theoretical range for the actual effective properties of heterogeneous materials. Even though both methods make simplifying assumptions about how stress and strain are spread out in the composite, they are still useful for initial analysis and for getting a basic understanding of how composite materials, like environmental concrete, behave physically and mechanically.

CHAPTER 3: MOONEY HOMOGENIZATION MODEL

INTRODUCTION TO CHAPTER 3

The Mooney homogenization model for concrete, first utilized in 1951 (Mooney, 1951), is employed to determine the effective elastic modulus of composite materials, including the reinforcement's spherical shape. The elastic modulus of the al mixture is depicted as an exponential function, featuring a parametric coefficient that signifies the impact of the reinforcement's volume fraction on the material's overall mechanical behavior. This formulation enables the model to illustrate the interrelationship between the effective stiffness of the composite, the amount of reinforcement, and the properties of the matrix.

Later, Einstein (1956) put forth another analytical model of homogenization in a fast state, predicated on the premise that reinforcing particles are rigidly included in a matrix medium at low concentration and are perfectly bonded to the surrounding matrix (Einstein, 1956). This model is based on ideas that were first put forward in earlier theoretical work. It offers a simpler way to explain how particulate media behave, especially in dilute suspensions where particle interactions are still limited.

Redzematovic and Kirane (2021) developed an innovative model for the homogenization of nonlinear graphene-based soft sandwich nanocomposites polymer plus the carbonaceous nature. They employed the rule of mixtures (ROM) and the Mooney–Rivlin constitutive model (Redzematovic and Kirane, 2021). Their research demonstrated that

nonlinear homogenization methods can address non-homogeneous problems, such as precisely capturing the nonlinear stress-strain response of nanocomposite materials, including Poisson effects and strain energy density, particularly when the reinforcement and matrix exhibit significantly disparate stiffness ratios.

Kim et al. (2007) also examined how various homogenization techniques influenced Mooney viscosity, specifically in the context of measuring natural rubber and conducting experimental processes. Their findings demonstrated that the homogenization method and rotor speed utilized during testing significantly influence the measured rheological properties and elasticity characteristics of the material, particularly at low rotational speeds. These results show how important homogenization procedures are for getting the mechanical and rheological properties of polymer-based composite materials right.

A lot of different materials, like composites and porous media, have a lot of small differences in them. In these cases, it becomes very hard to describe the material's complex microstructure. Homogenization techniques are employed to transform the irregular microstructure into a smooth macroscopic continuum that replicates the material's overall behavior (Auriault and Caillerie, 1989). There are many ways to make a material homogeneous, and the best method depends on the type of microstructure, whether it is periodic or random, and how well the characteristic scales of the microstructure are separated. A lot of researchers have studied periodic homogenization methods because

they can accurately predict how heterogeneous materials will act on a large scale.

This chapter talks about the Mooney homogenization method and how it can be used to figure out how well composite materials work mechanically. The model is especially useful for studying composites with particles that are spread out. It also gives a theoretical basis for figuring out how the volume fraction of reinforcement changes how heterogeneous materials respond mechanically as a whole.

3. MOONEY HOMOGENIZATION

3.1. Mechanical Homogenization with Mooney Model

In the Mooney homogenization model, the effective elastic modulus of the composite material is determined by considering the interaction between the reinforcement particles and the matrix, where the influence of the reinforcement volume fraction is represented through an exponential function within a representative volume element (RVE).

$$\frac{E}{E_m} = e^{\left(\frac{2.5 V_r}{1-s V_r}\right)} \quad \text{or} \quad E = E_m e^{\left(\frac{2.5 V_r}{1-s V_r}\right)}$$

(11)

Where:

$$1 < s < 2 \quad (12)$$

The total volume of the composite material is equal to the sum of the volume fractions of the reinforcement and the matrix, which is equal to unity, as indicated in Equation (2). The bulk modulus K can be expressed

as a function of Young's modulus E and Poisson's ratio ν for an isotropic elastic material, as given in Equation (13). The shear modulus G characterizes the material's resistance to shear deformation and is presented in Equation (14). The effective Poisson's ratio of the composite material is determined using a summation approach that accounts for the respective contributions of the reinforcement and the matrix according to their volume fractions, as shown in Equation (15).

$$G = \frac{E_m e^{\left(\frac{2.5 V_r}{1-s V_r}\right)}}{2(1 + \nu)} \quad (13)$$

$$K = \frac{E_m e^{\left(\frac{2.5 V_r}{1-s V_r}\right)}}{3(1 - 2\nu)} \quad (14)$$

$$\nu = V_r \nu_r + (1 - V_r) \nu_m \quad (15)$$

3.2. Thermal Homogenization with Mooney Model

Thermal homogenization provides an approach for estimating the effective thermal conductivity of composite materials by accounting for the thermal conductivities and volume fractions of their constituent phases.

$$\lambda = \lambda_m e^{\left(\frac{2.5 V_r}{1-s V_r}\right)} \quad (16)$$

The effective thermal expansion coefficient of the composite material is determined using a summation method that accounts for the contributions of the reinforcement and the matrix based on their volume fractions, as indicated in Equation (17).

$$\alpha = \alpha_m e^{\left(\frac{2.5 V_r}{1-s V_r}\right)} \quad (17)$$

3.3. Physical Homogenization with Mooney Model

Physical homogenization is used to evaluate the effective physical properties of composite materials by considering the properties and volume fractions of their constituent phases. The effective density of the composite material is determined as the weighted average of the densities of the reinforcement and the matrix according to their respective volume fractions, as indicated in Equation (18). Furthermore, the total mass of the composite material corresponds to the sum of the masses of the reinforcement and the matrix, as expressed in Equations (19) and (20).

$$\rho = \rho_m e^{\left(\frac{2.5 V_r}{1-s V_r}\right)} \quad (18)$$

$$m = \rho V \quad (19)$$

$$m = m_r + m_m \quad (20)$$

3.4. Exercises and Applications

Application 5:

Using the Mooney homogenization model, determine the effective mechanical properties of a concrete matrix ($E_m=20$ Gpa, $\nu_m=0.2$)

containing other type of glass powder as reinforcement ($E_r=50$ Gpa, $\nu_r=0.35$).

For reinforcement volume fractions V_r varying from 0% to 20%.

Table 5: The effective elasto-mechanical properties of new concrete (ordinary concrete + glass powder) when $s=1$.

V_r (%)	E (GPa)	ν	K (GPa)	G (GPa)
0	20.00	0.2000	11.11	8.33
5	22.81	0.2075	13.00	9.45
10	26.41	0.2150	15.44	10.87
15	31.09	0.2225	18.68	12.71
20	37.36	0.2300	23.06	15.18

The table 5 shows the evolution of the mechanical properties of the composite material as the reinforcement volume fraction V_r increases from 0% to 20%. As the reinforcement content increases, the elastic modulus (E) rises significantly from 20 GPa to 37.36 GPa, indicating a substantial improvement in the stiffness of the material. Similarly, the bulk modulus (K) increases from 11.11 GPa to 23.06 GPa, reflecting a higher resistance to volumetric deformation.

The shear modulus (G) also goes up steadily, from 8.33 GPa to 15.18 GPa. This means that the material is better able to resist shear deformation. The Poisson's ratio (ν) goes up a little bit, from 0.20 to

0.23, which means that the composite's lateral deformation behavior changes only a little.

Overall, the results show that adding more reinforcement makes the composite material stronger and stiffer. This trend shows that adding reinforcement particles to eco-friendly concrete or composite systems makes them work better.

Table 6: The effective elasto-mechanical properties of new concrete (ordinary concrete + glass powder) when $s=1.5$.

V_r (%)	E (GPa)	ν	K (GPa)	G (GPa)
0	20.00	0.2000	11.11	8.33
5	23.09	0.2075	13.16	9.57
10	27.35	0.2150	15.99	11.25
15	33.54	0.2225	20.14	13.71
20	43.23	0.2300	26.69	17.57

Table 7: The effective elasto-mechanical properties of new concrete (ordinary concrete + glass powder) when $s=2$.

V_r (%)	E (GPa)	ν	K (GPa)	G (GPa)
0	20.00	0.2000	11.11	8.33
5	23.26	0.2075	13.26	9.63
10	28.40	0.2150	16.61	11.68
15	37.74	0.2225	22.65	15.42
20	/	/	/	/

Tables 6 and 7 demonstrate how the new concrete's effective mechanical properties change for two distinct values of the parameter s ($s=1.5$ and $s=2$). As the reinforcement becomes stronger, the volume fraction r of the reinforcement climbs. Table 6 ($s = 1.5$) shows that increasing the reinforcing proportion from 0% to 20% makes the mechanical properties a lot better. To do this, the amount of reinforcement has to be raised. The material's stiffness rises up a lot when the elastic modulus (E) climbs from 20 GPa to 43.23 GPa. The bulk modulus (K) goes up from 11.11 GPa to 26.69 GPa, while the shear modulus (G) goes up from 8.33 GPa to 17.57 GPa. Both of these things happen at the same time. The Poisson's ratio (ν) just goes up a little bit, which means that the behavior of lateral deformation only changes a little bit.

Table 7 ($s = 2$) shows a similar trend for reinforcement ratios up to 15%. The elastic modulus goes up to 37.74 GPa and the shear modulus goes

up to 15.42 GPa. This shows that adding reinforcement makes the composite material stiffer, which is a good thing. On the other hand, the results for 20% reinforcement are not available, which might mean that the model has constraints or that the parameters that were chosen are not accurate.

The results show that adding more reinforcement to the new concrete makes it stiffer and less likely to bend. This shows how reinforcing may make environmentally friendly concrete materials work better structurally.

Application 6:

Use the Mooney homogenization model to find out how strong a concrete matrix is when it is reinforced with cardboard trash ($\lambda_r=1.65$ W/m \cdot °C). For reinforcement volume fractions V_r between 0% and 20%. Table 8 shows how the effective thermal conductivity (λ) of eco-concrete reinforced with cardboard (carton) waste changes for various reinforcement volume fractions (V_r) and three values of the parameter s (1, 1.5, and 2).

The findings reveal that when the reinforcement volume percentage goes from 0% to 20%, the thermal conductivity goes up. When $s=1$, for example, λ_m goes from 1.6500 W/m \cdot °C for plain concrete to 3.0826 W/m \cdot °C with 20% reinforcement. For $s=1.5$ and $s=2$, the thermal conductivity reaches 3.3705 W/m \cdot °C and 3.7966 W/m \cdot °C, respectively, which is a similar trend.

It is also clear that for the same reinforcement ratio, larger values of the parameters lead to higher thermal conductivity values. This means that the parameter s has a big effect on how the composite material behaves as it becomes hot.

The findings show that adding more cardboard waste to eco-concrete changes its thermal characteristics. They also show that both the reinforcing percent and the parameter s are significant for figuring out the material's effective thermal conductivity.

Table 8: The effective thermal properties who concrete reinforced with carton (cardboard) waste when parameters ($s=1,1.5,2$).

Vr (%)	λ (W/m*°C)		
	1	1.5	2
0	1.6500	1.6500	1.6500
5	1.8820	1.8887	1.8958
10	2.1783	2.2142	2.2553
15	2.5650	2.6769	2.8193
20	3.0826	3.3705	3.7966

3.5. Limitations of the Mooney Model

The Mooney homogenization model has certain problems that need to be fixed when compared to more classic approaches of homogenization, including the Voigt and Reuss models. One of the most important limits is the addition of an interaction parameter s , which usually needs to be found via study. The selection of this parameter may influence the precision of the projected effective characteristics. In contrast, the Voigt and Reuss models are simpler and don't need any additional factors. This is because their computations are based directly on the volume fractions of the different parts of the composite.

The Mooney model has certain problems, one of which is that mathematical singularities might happen when the denominator $1-X$ becomes closer and closer to zero. In these situations, the model could make predictions that are not true, particularly when the proportion of reinforcement volume is high. The Voigt and Reuss models show the best and worst possible effective properties of composite materials. This assures that the expected values are within the limits of what is physically possible.

The Mooney model also assumes that there is an exponential relationship between the effective modulus and the reinforcement %. This is a presumption that may not consistently represent the complex microstructure of heterogeneous composites. So, even though the Mooney model may show some impacts of particle interactions, it is usually only used for small amounts of reinforcement. The Voigt and

Reuss models, on the other hand, are nonetheless reliable reference points for figuring out the effective properties of materials that aren't the same.

3.6. Conclusion of chapter 3

The Mooney model calculates the effective characteristics of composite materials by taking into consideration how particles interact with each other via an exponential connection between the volume percent of the reinforcement and the stiffness of the composite. The Voigt and Reuss models, on the other hand, provide consistent upper and lower limits for the effective characteristics of composites. The Mooney model, on the other hand, may become unstable when the reinforcement volume percent is quite large. In certain situations, the model's mathematical formulation might make predictions about the material characteristics that aren't true. Because of this, the Mooney model may only be used for modest reinforcement amounts, usually between 20% and 30%. If the contents go over this range, the projected outcomes may not be as reliable.

CHAPTER 4: ESHELBY HOMOGENIZATION MODEL.

INTRODUCTION TO CHAPTER 4

The Eshelby model, introduced in 1957, is considered one of the most important analytical homogenization approaches used to determine the effective mechanical and physical properties of composite materials (Eshelby, 1957). It provides an exact analytical solution for the problem of ellipsoidal or spherical inclusions embedded in an infinite elastic matrix, making it particularly suitable for the analysis of heterogeneous materials. This model is especially applicable to two-phase composite systems (matrix–inclusion) where the inclusions are assumed to be spherical or ellipsoidal. In addition, the model generally assumes that the inclusions are dilute distributed and that their volume fraction remains relatively small so that the interactions between inclusions can be neglected.

Under these assumptions, the Eshelby model enables the determination of the stress and strain fields both inside and around the inclusions, which makes it a powerful theoretical framework for estimating the effective properties of composite materials. The analytical formulation proposed by Eshelby has therefore become a fundamental basis for many subsequent homogenization models used in composite mechanics.

Several researchers have contributed to the development and application of this model. For instance, the PhD thesis of Bourih (2017) proposed several analytical developments and proportional solutions

related to the classical inclusion problem, contributing to improvements in homogenization techniques for heterogeneous materials (Bourih, 2017).

More recently, Benfrid et al. (2023) performed a thorough thermomechanical analysis of concrete reinforced with glass powder at the nanoscale, utilizing the Eshelby homogenization model. The researchers hypothesized that the glass particles functioned as spherical inclusions within the concrete matrix, considered as reinforcement in an isotropic condition. The research integrated homogenization methodologies with higher-order shear deformation plate theory to examine reinforced concrete panels under thermal loading conditions. The findings yielded novel numerical data that can function as valuable reference points for subsequent investigations in eco-concrete and thermomechanical applications (Benfrid et al., 2023).

In order to determine the strength of concrete slabs, Kecir et al. (2023) also carried out a multi-phase assessment of their mechanical characteristics. impact of ferrous metal nanoparticles (Fe_2O_3). About concrete slabs' mechanical flexural strength. The Eshelby method was used by the researchers to assess the tensile and flexural properties of nanocomposite materials made of concrete and nanosteel. a model for achieving consistency in small-scale nanotechnology. Research indicates that metal nanoparticles can be made significantly more flexible and less prone to bend by adding roughly 30% of the material's weight. over 60% of sheets of thin concrete. As a result, nanotechnology

is very promising. in order to increase the stability of concrete structures (Kecir et al., 2023).

In a related study, Chatbi et al. (2023) examined the static behavior of concrete beams reinforced with clay nano-platelets using a quasi-3D beam theory. The Eshelby homogenization model was used to evaluate the equivalent elastic properties of the nanocomposite material. Their results demonstrated that the mechanical strength and structural performance of concrete beams can be significantly increased by adding nano-platelets, offering promising prospects for the development of advanced reinforced concrete materials (Chatbi et al., 2023).

This chapter presents the Eshelby homogenization model, which provides an analytical solution for the problem of spherical or ellipsoidal inclusions embedded within an elastic matrix. For two-phase composite materials with diluted inclusion distributions and low reinforcement volume fractions, the model performs particularly well. By examining the stress and strain fields inside and around the inclusions, the model enables the determination of the effective mechanical properties of heterogeneous materials.

This chapter also discusses the basic assumptions of the Eshelby model and how they relate to composite materials, such as environmental concrete reinforced with different inclusions. Finally, the advantages and disadvantages of the Eshelby approach are briefly examined in comparison to other conventional homogenization models used in composite material analysis.

4. ESHELBY HOMOGENIZATION

4.1. Eshelby Model to predict electromechanically factors

The Eshelby model is a mathematical way to talk about how composite materials become more uniform. It is based on the idea of elasticity and is used to predict how materials will act when they are under various kinds of stress and strain. The British physicist John D. Eshelby came up with this idea in the 1950s. The Eshelby model is made up of three basic parts: the stress field, the strain field, and the elastic modulus. The strain field is a mathematical way to describe how a material changes shape when a certain force is put on it. The stress field shows how the pressures are spread out across the material. The elastic modulus, on the other hand, shows how rigid the material is. Using the Eshelby model, you can properly guess the stress and strain fields within composite materials that include inclusions. This helps us understand how materials behave mechanically when they are under various types of stress, including thermomechanical stresses or fatigue loading. This method is thus particularly beneficial for designing and improving structural parts that can handle heavy loads. Eshelby also pointed out an essential use for these conclusions in the study of inclusions that are embedded in elastic matrices. This is the foundation for many recent homogenization techniques used in composite materials.

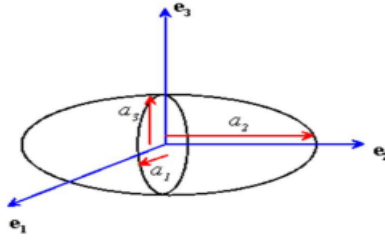


Figure 9: Eshelby inclusion (ellipsoidal inclusion).

Consider an infinite, homogeneous, isotropic, and linearly elastic solid in which a uniform eigenstrain is introduced within an ellipsoidal region.

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq 1 \quad (21)$$

We may apply the procedure described in the previous section to determine the fields within the solid. In the present case, the body force is zero everywhere except on the surface of the ellipsoid, where it exhibits a singularity. Here, n denotes the outward normal to the ellipsoid. The potential integrals are exact only for a spherical region, but they can be reduced to elliptic integrals for the ellipsoidal case. Inside the ellipsoid, the stress and strain fields are uniform, whereas outside the ellipsoid they are more complex, although they can still be expressed in terms of a few elliptic integrals. In this section, only the results for the ellipsoidal region are presented, where the total strain can

be written in a simplified form.

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + \varepsilon_{ij}^i = S_{ijkl} \varepsilon_{kl}^* \quad (22)$$

The Eshelby tensor depends on the elastic properties of the solid and on the semi-axes of the ellipsoidal inclusion. Although its components are generally complex, they are presented here for the case of an isotropic elastic solid.

$$\begin{aligned} S_{1111}^* &= \frac{3}{8\pi(1-\nu)} a_1^2 I_{11} - \frac{1-2\nu}{8\pi(1-\nu)} I_1 \\ S_{1122}^* &= \frac{3}{8\pi(1-\nu)} a_2^2 I_{12} - \frac{1-2\nu}{8\pi(1-\nu)} I_1 \\ S_{1133}^* &= \frac{3}{8\pi(1-\nu)} a_3^2 I_{13} - \frac{1-2\nu}{8\pi(1-\nu)} I_1 \\ S_{1212}^* &= \frac{a_1^2 + a_2^2}{16\pi(1-\nu)} I_{12} - \frac{1-2\nu}{16\pi(1-\nu)} (I_1 + I_2) \\ S_{1112}^* &= S_{1223}^* = S_{1232}^* = 0 \end{aligned} \quad (23)$$

$$\text{In wich: } a_1 > a_2 > a_3 \quad (24)$$

We written :

$$\begin{aligned}
I_1 &= \frac{4\pi a_1 a_2 a_3}{(a_1^2 - a_2^2)(a_1^2 - a_3^2)} F(\theta, k) \\
I_3 &= \frac{4\pi a_1 a_2 a_3}{(a_2^2 - a_3^2)(a_1^2 - a_3^2)} \left[\frac{a_2 \sqrt{a_1^2 - a_2^2}}{a_1 a_3} - E(\theta, k) \right]
\end{aligned}
\tag{25}$$

When, The parameters θ and k are introduced to express the elliptic integrals involved in the calculation of the Eshelby tensor. They depend solely on the geometric parameters of the ellipsoidal inclusion.

$$\theta = \sin^{-1} \left(\sqrt{1 - \frac{a_3^2}{a_1^2}} \right) \quad k^2 = \frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}
\tag{26}$$

These relations establish the connections between the elliptic integrals involved in the evaluation of the Eshelby tensor components for an ellipsoidal inclusion embedded in an isotropic elastic matrix.

$$\begin{aligned}
I_1 + I_2 + I_3 &= 4\pi & 3a_1^2 I_{11} + a_2^2 I_{12} + a_3^2 I_{13} &= 3I_1 \\
3I_{11} + I_{12} + I_{13} &= \frac{4\pi}{a_1^2} & I_{12} &= \frac{I_2 - I_1}{a_1^2 - a_2^2}
\end{aligned}
\tag{27}$$

Outside the ellipsoid, the strain fields are non-uniform but can be expressed in terms of three elliptic integrals. A comprehensive derivation of the theoretical framework associated with this model is presented in *Micromechanics of Defects in Solids* (Mura, 1982). For the

special case of a spherical inclusion, these results become significantly simpler.

$$S_{1111}^* = S_{2222}^* = S_{3333}^* = \frac{7 - 5\nu}{15(1 - \nu)}$$

$$S_{1122}^* = S_{2233}^* = S_{3311}^* = S_{1133}^* = S_{2211}^* = S_{3322}^* = \frac{5\nu - 1}{15(1 - \nu)}$$

$$S_{1212}^* = S_{2323}^* = S_{3131}^* = \frac{4 - 5\nu}{15(1 - \nu)} \quad (28)$$

The effective stiffness tensor of the composite material can be obtained using the Eshelby homogenization framework, which accounts for the elastic properties of both the matrix and the inclusions, as well as the geometric influence described by the Eshelby tensor.

$$C = \left[C_m^{-1} - V_r \{ (C_m - C_r) [S - V_r(S - I)] + C_m \}^{-1} (C_r - C_m) C_m^{-1} \right]^{-1} \quad (29)$$

After simplification, the effective bulk, shear, and Young's moduli of the composite can be obtained from the Eshelby homogenization model by replacing the inclusion and matrix parameters with those of the reinforcement and matrix phases.

$$K = K_m + V_r \frac{(K_r - K_m)(3K_m + 4K_r)}{3K_r + 4K_m}$$

$$G = G_m \left[1 + \frac{15V_r \left(1 - \frac{G_r}{G_m} \right) (1 - \nu_m)}{7 - 5\nu_m + 2(4 - 5\nu_m) \frac{G_r}{G_m}} \right]$$

$$E = \frac{9K}{1 + 3K G^{-1}} \quad (30)$$

4.2. Eshelby Model to evaluate thermal properties

Researchers were able to anticipate that the effective thermal conductivity of a composite material with spherical inclusions could be expressed as a function of the thermal conductivities of the matrix and reinforcement, along with their respective volume fractions. This was made possible by the Eshelby homogenization technique utilized by the researchers.

$$\lambda = \lambda_m \left[1 + \frac{3V_r(\lambda_r - \lambda_m)}{\lambda_r + 2\lambda_m - V_r(\lambda_r - \lambda_m)} \right] \quad (31)$$

The effective coefficient of thermal expansion of the composite can be determined using the Eshelby homogenization model by accounting for the elastic properties of the matrix and the inclusions as well as the influence of the inclusion geometry through the Eshelby tensor.

$$\alpha = \alpha_m - V_r \{ (C_m - C_r) [S - V_r(S - I)] - C_m \}^{-1} C_r (\alpha_r - \alpha_m) \quad (32)$$

4.3. Physical Homogenization with Eshelby Model

The effective density of the composite material is determined as the weighted average of the densities of the reinforcement and the matrix according to their respective volume fractions, while the total mass is equal to the sum of the masses of the two phases.

$$\rho = V_r \rho_r + (1 - V_r) \rho_m \quad (33)$$

$$m = \rho V \quad (34)$$

$$m = m_r + m_m \quad (35)$$

4.4. Exercises and Applications

Application 7:

Apply the Eshelby homogenization model to evaluate the effective mechanical properties of a concrete matrix ($E_m=20$ Gpa , $\nu_m=0.2$) containing porcelain particles as reinforcement ($E_r=60$ Gpa , $\nu_r=0.2$).

Considering reinforcement volume fractions V_r varying between 0% and 30%.

Table 9: The effective mechanical properties of concrete reinforced with porcelain particles.

V_r (%)	E (GPa)	ν	K (GPa)	G (GPa)
0	20.00	0.20	11.11	8.33
5	22.00	0.20	12.22	9.17
10	24.00	0.20	13.33	10.00
15	26.00	0.20	14.44	10.83
20	28.00	0.20	15.56	11.67
25	30.00	0.20	16.67	12.50
30	32.00	0.20	17.78	13.33

Table 9 shows how the effective mechanical characteristics of concrete containing porcelain particles as reinforcement change as the volume fraction of the reinforcement V_r goes from 0% to 30%.

The findings reveal that the elastic modulus (E) goes up from 20 GPa to 32 GPa over time. This means that adding porcelain particles made the composite material more stiffer.

The bulk modulus (K) also goes up, from 11.11 GPa to 17.78 GPa, which means that the material is less likely to change shape when it is compressed. The shear modulus (G) goes up from 8.33 GPa to 13.33 GPa, which means that the material is better able to withstand shear deformation.

It is also clear that the Poisson's ratio (ν) stays the same at 0.20 for all reinforcement ratios. This means that adding porcelain particles doesn't have a big effect on how the composite deforms sideways.

The findings show that adding more porcelain particles makes the reinforced concrete noticeably stiffer and more rigid. This shows that porcelain waste might be used as a reinforcing element in environmentally friendly building projects.

Application 8:

Use the Eshelby homogenization model to get the effective thermal and physical properties of a standard concrete matrix with a temperature of $1.65 \text{ W/m} \cdot ^\circ\text{C}$ and a temperature of $10 \times 10^{-6} \text{ } ^\circ\text{C}^{-1}$. Cardboard trash, which has a temperature of $0.037 \text{ W/m} \cdot ^\circ\text{C}$, is used to strengthen the matrix. $^\circ\text{C}$ and a temperature of $9 \times 10^{-6} \text{ } ^\circ\text{C}^{-1}$. This group includes reinforcement volume percentages from 0% to 30%. Table 10 shows how the effective thermal properties of eco-concrete that has been strengthened with cardboard (carton) waste have changed. This change happens when the reinforcing volume fraction V_p goes from 0% to 30%. The results reveal that the thermal conductivity (λ) drops continuously from $1.650 \text{ W/m} \cdot ^\circ\text{C}$ to $1.075 \text{ W/m} \cdot ^\circ\text{C}$ as the amount of cardboard increases. The fact that this number went down shows that adding old cardboard to eco-concrete makes it better at keeping heat in. There may be less heat transmission within the composite material because of particles made of cellulose and pores in the structure of the cardboard.

The coefficient of thermal expansion (α) also drops somewhat, from 10.0 C^{-1} to 9.70 C^{-1} . This means that the material is a little more stable when the temperature changes.

The results as a whole show that adding more cardboard waste to eco-concrete makes it work better thermally by reducing both its thermal conductivity and its thermal expansion. This shows how recycled cardboard trash might be an eco-friendly way to improve the energy efficiency of concrete materials in construction projects that are sustainable.

Table 10: The effective thermal properties of eco concrete plus carton (cardboard) waste.

Vr (%)	λ (W/m*°C)	$\alpha \times 10^{-6}$ (C ⁻¹)
0	1.650	10.0
5	1.535	9.95
10	1.428	9.90
15	1.330	9.85
20	1.238	9.80
25	1.154	9.75
30	1.075	9.70

4.5. Limitations of the Eshelby Model

The Eshelby homogenization model has several limitations when it comes to composite materials that might be seen as issues. The model has a lot of assumptions, but one of the most essential is that the inclusions are ellipsoidal and are set in a matrix that is infinite, homogeneous, and isotropic. These traits may not always accurately show the complex microstructures of real composite materials. The model is also only useful for inclusion volume fractions that are low or moderate, which means that the interactions between the inclusions may be neglected. When there is a lot of reinforcement, the interactions between particles become quite strong. This might make the model's predictions less accurate.

The Eshelby model also assumes that the matrix and the inclusions are completely bonded to each other. In real-world materials, the interface between phases might include flaws, debonding, or inadequate adhesion. Any of these issues can have a big effect on how well the composite works as a whole. Thus, while the Eshelby model provides a substantial theoretical foundation for assessing the effective properties of composite materials, its use is limited to scenarios where these simplifying assumptions are well satisfied.

4.6. Conclusion of chapter 4

When used on composite materials, the Eshelby homogenization model has a number of problems. The model implies that the inclusions are ellipsoidal in form and are embedded in a matrix that is infinite,

homogenous, and isotropic. These assumptions may not be true for the complicated microstructure of genuine composite materials. Also, the model is only really useful for low or moderate inclusion volume fractions, when the effects of interactions between inclusions may be ignored. When the amount of reinforcement is substantial, the interactions between particles become important, which might make the projected effective qualities less accurate.

A key part of the Eshelby model is that the matrix and the inclusions are perfectly bonded to each other. In real-world materials, however, the interface between the two phases may include flaws, poor adhesion, or partial debonding, which may have a big effect on how the composite behaves mechanically. So, although while the Eshelby model is a useful way to think about how to estimate the effective qualities of materials that aren't all the same, it can only be used when the assumptions it makes are mostly true.

CHAPTER 5: HASHIN–SHTRIKMAN HOMOGENIZATION MODEL.

INTRODUCTION TO CHAPTER 5

Composite materials are commonly utilized in engineering because they may combine the best qualities of diverse parts. It is still hard to predict the effective elastic and thermal characteristics of these kinds of mixed materials in material science and solid mechanics. Homogenization techniques are very important for figuring out how multiphase materials would behave on a large scale based on the characteristics of their small parts.

The Hashin–Shtrikman model, created by Hashin and Shtrikman in 1963, is one of the most important contributions to this discipline. This model presupposes that a two-phase composite material functions as an isotropic medium, indicating that its elastic characteristics are uniform in all spatial directions. In this paradigm, the upper and lower Hashin–Shtrikman bounds are two theoretical limits that set the effective elastic modulus of the composite. Using the elastic polarization tensor (Hashin & Shtrikman, 1963) and the linear theory of elasticity, we may get these limits.

The Hashin–Shtrikman method gives very accurate estimations of the effective elastic moduli of multiphase materials that are quasi-isotropic and quasi-homogeneous, no matter how their phases are arranged in space. When the difference between the elastic properties of the different phases is not too big, the anticipated upper and lower limits

become near enough to provide good estimates of the effective mechanical characteristics (Hashin & Shtrikman, 1963).

There have been a number of research, both theoretical and practical, carried out in order to validate the limits that have been established. As an example, Watt and O'Connell (1980) used ultrasonic interferometry data in order to investigate the elastic properties of two-phase aggregates. According to Watt and O'Connell's research from 1980, the results demonstrated that the compressional and shear wave velocities obtained from composite materials were in agreement with the theoretical Hashin–Shtrikman limitations. This was the case even after taking into account modifications for anisotropy and porosity.

The Hashin–Shtrikman framework was enlarged to encompass more complex mechanical situations as a result of subsequent developments. Luciano and Willis (2002) provided an illustration of the influence that non-local interactions have on heterogeneous materials when the structural dimensions are getting closer to the size of the microstructure. According to the findings of their study, boundary-layer effects and stress–strain concentration phenomena have the potential to significantly modify the macroscopic behavior of composite materials (Luciano & Willis, 2002).

In the field of cement-based goods, micromechanical techniques have also been used rather extensively in order to determine the elastic modulus of concrete. Barbosa et al. (2011) investigated the development of the elastic modulus by studying concrete as a three-phase composite that included aggregates, cement paste, and the

interfacial transition zone (ITZ). This allowed them to evaluate the evolution of the elastic modulus. The results of their research showed that micromechanical models, such as the Mori–Tanaka and three-phase sphere models, are able to provide predictions that are in good agreement with experimental data when the interfacial transition zone (ITZ) is taken into consideration in the right manner (Barbosa et al., 2011).

An investigation on the impact that glass nanoparticles have on the elastic modulus of concrete was carried out by Benfrid et al. (2024). Within the scope of their investigation, they compared the theoretical predictions that were produced from homogenization models with the actual results for volume fractions ranging from 0% to 30%. Benfrid et al. (2024) reported that the findings indicated that the Hashin–Shtrikman model is capable of accurately predicting the mechanical properties of nano-glass-reinforced eco-concrete. This highlights the model's utility in promoting sustainable construction practices through the recycling of glass waste and minimizing the necessity for extensive experimental validation.

The primary objective of Chapter 5, which is titled "Hashin–Shtrikman Homogenization Model," is to provide an explanation of the theoretical framework of the Hashin–Shtrikman methodology and to implement it in the process of mechanical and thermal homogenization of concrete reinforced with waste materials, which is considered to be a biphasic composite material. In this chapter, the concrete matrix and waste particles are modeled as two distinct phases. The Hashin–Shtrikman

homogenization approach is used in order to determine the effective elastic and thermal parameters of the composite material. Next, we will study the theoretical predictions in order to get a better understanding of how the incorporation of rubbish influences the overall behavior of the eco-concrete that we have been investigating.

5. HASHIN–SHTRIKMAN HOMOGENIZATION

5.1. Mechanical Homogenization with Hashin–Shtrikman Model

The Hashin–Shtrikman model provides theoretical upper and lower bounds for the effective bulk modulus of a two-phase composite material. These bounds depend on the bulk and shear moduli of the constituent phases as well as their volume fractions.

$$\begin{aligned}
 K_{HS}^+ &= K_r + \frac{V_m}{\frac{1}{K_m - K_r} + \frac{V_r}{K_r + \frac{4}{3}G_r}} \\
 K_{HS}^- &= K_m + \frac{V_r}{\frac{1}{K_r - K_m} + \frac{V_m}{K_m + \frac{4}{3}G_m}}
 \end{aligned} \tag{36}$$

The effective shear modulus of the composite is also bounded by the Hashin–Shtrikman limits.

$$\begin{aligned}
 G_{HS}^+ &= G_r + \frac{V_m}{\frac{1}{G_m - G_r} + \frac{6V_r(K_r + 2G_r)}{5G_r(3K_r + 4G_r)}} \\
 G_{HS}^- &= G_m + \frac{V_r}{\frac{1}{G_r - G_m} + \frac{6V_m(K_m + 2G_m)}{5G_m(3K_m + 4G_m)}}
 \end{aligned} \tag{37}$$

The effective Young's modulus of the equivalent homogeneous composite material is given by equation (38).

$$E = \frac{9KG}{3K + G} \quad (38)$$

The effective Poisson's ratio of the composite material can be calculated as equation (39)

$$\nu = \frac{3K - 2G}{2(3K + G)} \quad (39)$$

5.2. Thermal Homogenization with Hashin–Shtrikman Model

Thermal homogenization is an important step in predicting the effective thermal behavior of heterogeneous composite materials. In the case of a biphasic composite such as concrete reinforced with waste particles, the overall thermal conductivity depends on the thermal conductivities of the constituent phases as well as their respective volume fractions. Homogenization models make it possible to estimate the equivalent thermal conductivity of the composite material by considering it as an effective homogeneous medium.

Among the available theoretical approaches, the Hashin–Shtrikman model provides rigorous upper and lower bounds for the effective thermal conductivity of two-phase composite materials. These bounds are derived using variational principles and are considered among the

most accurate theoretical limits for heterogeneous media (Hashin & Shtrikman, 1963).

In the present study, the composite material is considered as a biphasic medium composed of a concrete matrix and waste reinforcement particles. The volume fractions of the two phases are denoted by V_m for the matrix and V_r for the reinforcement, such that:

$$V_m + V_r = 1 \quad (40)$$

$$\lambda_{HS}^+ = \lambda_r + \frac{V_m}{\frac{1}{\lambda_m - \lambda_r} + \frac{V_r}{3\lambda_r}} \quad \lambda_{HS}^- = \lambda_m + \frac{V_r}{\frac{1}{\lambda_r - \lambda_m} + \frac{V_m}{3\lambda_m}} \quad (41)$$

5.3. Physical Homogenization with Hashin–Shtrikman Model

In addition to the mechanical and thermal properties, the density of the composite material is also an important parameter in the homogenization analysis.

$$\rho = V_r \rho_r + (1 - V_r) \rho_m \quad (42)$$

$$m = \rho V \quad (43)$$

When: $m = m_r + m_m$ (44)

5.4. Exercises and Applications

Application 9:

Apply the Hashin–Shtrikman homogenization model to evaluate the effective mechanical properties of a concrete matrix ($E_m=20$ GPa, $\nu_m=0.2$) containing porcelain particles as reinforcement ($E_r=60$ GPa, $\nu_r=0.2$).

Considering reinforcement volume fractions V_r varying between 0% and 30%.

Table 11: The effective mechanical properties of concrete reinforced with porcelain particles.

V_r (%)	K inf (GPa)	K sup (GPa)	G inf (GPa)	G sup (GPa)	E inf (GPa)	E sup (GPa)
0	11.111	11.111	8.333	8.333	20	20
5	11.681	11.864	8.761	8.898	21.026	21.356
10	12.281	12.644	9.211	9.483	22.105	22.759
15	12.913	13.45	9.685	10.088	23.243	24.211
20	13.58	14.286	10.185	10.714	24.444	25.714
25	14.286	15.152	10.714	11.364	25.714	27.273
30	15.033	16.049	11.275	12.037	27.059	28.889

The Hashin–Shtrikman homogenization model shows that the volume % of the reinforcement has a clear impact on the composite material's effective mechanical properties. As the volume of porcelain particles in the composite material goes from 0% to 30%, the effective Young's

modulus of the material rises quickly. We expect this to happen in the reinforcement phase since it has a higher elastic modulus ($E_r=60\text{GPa}$) than the concrete matrix ($E_m=20\text{GPa}$). Adding porcelain particles to the composite material makes it stiffer in all directions.

The Hashin–Shtrikman limitations show the range of useful mechanical properties that the composite material would need to have. For the purposes of this study, the difference between the top and lower bounds stays rather small throughout the different reinforcement volume percentages that were looked at. This shows that the expected effective characteristics are correct and that the model performs a great job of showing how the composite material will work mechanically over time. The effective Poisson's ratio of the composite stays almost the same from one reinforcement volume fraction to the next over the range of reinforcement volume fractions that were studied. This is because both the matrix and reinforcement phases share the same Poisson's ratio ($\nu=0.2$). Adding porcelain particles usually changes the stiffness of the material, but it doesn't change the shape of the material much when it's compressed. The results suggest that adding leftover porcelain to concrete could make it stronger. This sort of waste not only makes things stronger, but it also helps the environment by encouraging the recycling of industrial ceramic waste and lowering the negative effects that regular building materials have on the environment. It improves the environment, in other terms. The Hashin–Shtrikman homogenization model is a useful theoretical tool for figuring out the effective mechanical properties of biphasic composites, such concrete that has

rubbish in it. The model shows you how the volume % of reinforcement influences the scenario without needing you to undertake a lot of trials. Because of this, it's a quick way to start the process of making and improving materials.

Application 10:

Using the Hashin–Shtrikman homogenization model, find the effective thermal and physical parameters of a regular concrete matrix ($\lambda_m=1.65$ W/m*°C, $\alpha_m=10\times 10^{-6}$ °C⁻¹) that has been strengthened with cardboard waste ($\lambda_r=0.037$ W/m*°C, $\alpha_r=9\times 10^{-6}$ °C⁻¹). For reinforcement volume fractions V_r between 0% and 30%.

It was noted that the thermal expansion coefficient only goes down a little bit when the amount of reinforcement goes up since the thermal expansion coefficients of the two phases are very similar. These studies show that cardboard waste may be a good way to make eco-friendly concrete that has better thermal insulation and yet expands and contracts in a steady way.

Table 12: The effective thermal properties of eco concrete based carton (cardboard) waste.

V_r (%)	λ inf (W/m·°C)	λ sup (W/m·°C)	α×10⁻⁶ (C⁻¹)
0	1.65	1.65	10
5	0.925	1.533	9.95
10	0.629	1.422	9.9
15	0.468	1.315	9.85
20	0.367	1.214	9.8
25	0.298	1.116	9.75
30	0.248	1.023	9.7

5.5. Limitations of the Hashin–Shtrikman Model

People typically utilize the Hashin–Shtrikman (HS) homogenization model to anticipate what valuable qualities composite materials will have. But it's important to remember that this model has several problems that should be kept in mind while looking at the outcomes.

The HS model presumes that the composite material is an isotropic and homogenous medium, even when seen at a macro size. In actuality, concrete that has been fortified with waste particles may contain microstructures that are not the same in all directions and are not uniform. This is particularly true when the particles are not spherical or are not equally spaced out.

The second issue with the model is that it doesn't offer a definite value for the effective characteristics; it simply gives theoretical upper and lower limits. The actual mechanical or thermal characteristics of the composite material will fall between these two boundaries. The arrangement of the microstructure and the way the phases interact will determine where these characteristics exist.

One other problem with the HS model is that it assumes that the matrix phase and the reinforcement phase operate flawlessly together. The interfacial transition zone (ITZ) between the matrix and the inclusions may have a substantial influence on the overall characteristics of a composite, such as waste-reinforced concrete, that is helpful in real life. The HS formulation does not explicitly include variables like porosity, microcracking, and weak bonding.

The model was also built for composite materials that only have two dimensions. A basic biphasic approximation cannot adequately represent the microstructures of real concrete, which may consist of particles, porosity, cement paste, and other phases. This approximation doesn't adequately describe these microstructures.

Lastly, the Hashin–Shtrikman model doesn't take into account size effects, particle clustering, or non-local interactions. These are all things that may happen when the microstructure is about the same size as the structure.

The Hashin–Shtrikman model is still an important theoretical tool, even though it has known flaws. This is because it helps us make accurate

estimates of the effective properties of composite materials and looks at how the volume percentage of reinforcement affects biphasic systems like waste-reinforced concrete.

5.6. Conclusion of chapter 5

This chapter presented the Hashin–Shtrikman homogenization model as a crucial theoretical framework for assessing the effective properties of heterogeneous composite materials. This chapter introduced the model. In the subsequent chapter, we will elaborate on this concept in greater depth. This model establishes stringent top and lower bounds for the functional mechanical and thermal characteristics of biphasic materials. The model establishes these boundaries. These limits are determined by the intrinsic qualities of the material and the volume fractions of its two phases. This section discussed the intended functionality of the model. This formulation's equations illustrate the density, thermal conductivity, Young's modulus, Poisson's ratio, shear modulus, and bulk modulus of a homogeneous medium. These links enable the prediction of the overall behavior of composite materials by examining the simultaneous interactions between the matrix phase and the reinforcement phase.

The Hashin–Shtrikman constraints stipulate that the actual effective qualities of the composite must reside within a predetermined range of values. This method is effective for determining the influence of reinforcing volume on various factors and how the combination of different phases may alter the overall thermo-mechanical properties of

composite materials. The model was constructed using fundamental concepts such as isotropy, ideal phase interfaces, and a biphasic structure. Regardless of the circumstances, it serves as an invaluable tool for micromechanics and material uniformity. The Hashin–Shtrikman model serves as an effective framework for predicting the properties of composite materials prior to empirical testing.

CHAPTER 6: MORI–TANAKA HOMOGENIZATION MODEL.

INTRODUCTION TO CHAPTER 6

The Mori–Tanaka homogenization model was developed by Mori and Tanaka in 1973. They sought to determine the factors contributing to the efficacy of composite materials featuring scattered inclusions within a continuous matrix. It has been determined that estimating the internal stress field in a material with inclusions experiencing transformation strain is doable by determining the average stress in the matrix. They established that the mean stress within the matrix remains constant irrespective of the averaging domain's position, ensuring homogeneity across the material. The real stress in the matrix can be expressed as the sum of the average stress and a variable local stress, the average of which approaches zero throughout the matrix. Numerous methods exist to articulate the actual tension within the matrix. The model enables the calculation of the average elastic energy by considering the interactions among the inclusions. This is a crucial component for materials enhanced through dispersion techniques (Mori & Tanaka, 1973).

Recent research have employed the Mori–Tanaka model to investigate the mechanical characteristics of composite materials containing nanostructured or recycled inclusions. Boukari et al. (2025) investigated the bending characteristics of a concrete beam reinforced with granite-derived waste powder. The Mori–Tanaka homogenization model was employed to assess the performance of the composite material with varying quantities of granite powder. Increasing the

reinforcement ratio was demonstrated to reduce beam deflection while simultaneously enhancing the elastic modulus, shear resistance, and compressibility. The research confirmed this assertion.

In a similar vein, Yakro et al. (2025) examined the thermo-mechanical buckling characteristics of concrete panels that integrate waste elements, including glass and red brick. The Mori–Tanaka model was employed to ascertain the effective thermo-mechanical characteristics of the composite prior to evaluating the critical buckling load and temperature. Their findings indicated that the integration of waste materials enhances the mechanical qualities of concrete, but potentially diminishing certain thermal characteristics.

The Mori–Tanaka homogenization method has been employed to assess the mechanical properties of advanced nanocomposites. Melati et al. (2025) utilized this model to assess the effective elastic characteristics of nanocomposite plates reinforced with a steel–nano-tungsten alloy. Their analytical study demonstrated that the model yields precise predictions for stress and displacement distributions in composite plates under flexural loading.

Khetir et al. (2026) investigated the mechanical properties of lightweight gypsum concrete augmented with bio-sourced nanoparticles obtained from bovine bones and plant fibers. Employing the Mori–Tanaka model, they illustrated that the integration of these nanoparticles markedly improves the bending performance and mechanical strength of lightweight composite beams.

Recently, Benfrid (2026) examined the mechanical properties of concrete strengthened using nano-inclusions sourced from aluminum waste. This study employed the Mori–Tanaka homogenization model to assess the effective elastic characteristics of a composite material with spherical nano-inclusions. The findings demonstrated that the elastic modulus, shear modulus, and bulk modulus augment with the inclusion content. Furthermore, the model offers researchers a pragmatic instrument for forecasting appropriate reinforcement volume fractions, hence minimizing experimental endeavors. To ensure safety and material stability, the maximum reinforcing fraction was restricted to 30%.

Recent studies in sustainable construction materials indicate an increasing focus on employing natural resources and industrial byproducts to improve mechanical performance and environmental sustainability. The example demonstrates that integrating zeolite tuff and brown trash in the manufacture of ceramic bricks can yield eco-friendly products. While preserving adequate physical qualities (Ibrahim et al., 2023; Şahin et al., 2023). Moreover, geopolymer foam derived from silica fume and natural zeolite tuff significantly enhances the insulating qualities of For. Specifically by the utilization of foaming chemicals like aluminum and hydrogen peroxide. Which significantly influence pore structure and mechanical properties (Ibrahim et al., 2024; Şahin & Ibrahim, 2025). Enhancing energy efficiency and mitigating carbon dioxide emissions are essential objectives for

attaining sustainable development within the construction sector (Tırnk et al. 2025, Argun et al., 2026).

The primary objective of Chapter 6 is to demonstrate the construction of the Mori–Tanaka model and subsequently apply it to determine the effective mechanical, thermal, and physical properties of a biphasic concrete composite containing waste materials. This is predicated on the Mori–Tanaka homogenization technique. This chapter aims to examine the impact of reinforcing volume percentage on the overall performance of composite materials and the feasibility of utilizing waste to enhance the efficacy of eco-friendly construction materials.

6. MORI–TANAKA HOMOGENIZATION

6.1. Mori–Tanaka Model to determined elastic's properties

The Mori–Tanaka homogenization method, formulated by Mori and Tanaka in 1973, is frequently employed to evaluate the effective elastic properties of composite materials containing dispersed inclusions within a continuous matrix, this technique can predict mechanical and thermal properties. This methodology is especially suitable for heterogeneous materials like concrete, wherein the cementitious phase functions as the continuous matrix and reinforcing particles are interspersed throughout the material (Mori and Tanaka, 1973).

In the present case, the composite material consists of a concrete matrix reinforced with waste particles acting as inclusions. The homogenization analysis is carried out by considering a representative volume element (RVE) that contains both the matrix and the inclusions.

The total volume of the composite material is therefore expressed as equation number (45):

$$V_m + V_r = 1 \quad (45)$$

Using the Mori–Tanaka model, the effective bulk modulus K and effective shear modulus G of the composite material are given by flowing equations (46) to (48) :

$$K = K_m + \frac{V_r(K_r - K_m)}{1 + (1 - V_r)\frac{K_r - K_m}{K_m + \frac{4}{3}G_m}} \quad (46)$$

$$G = G_m + \frac{V_r(G_r - G_m)}{1 + (1 - V_r)\frac{G_r - G_m}{G_m + f}} \quad (47)$$

When:

$$f = \frac{G_m(9K_m + 8G_m)}{6(K_m + 2G_m)} \quad (48)$$

Once the effective bulk and shear moduli are determined, the effective Young's modulus E and Poisson's ratio ν of the composite can be calculated using the classical elasticity relations number (49) and (50).

$$E = \frac{9KG}{3K + G} \quad (49)$$

$$\nu = \frac{3K - 2G}{2(3K + G)} \quad (50)$$

6.2. Thermal Homogenization with Mori–Tanaka Model

The effective thermal conductivity of the composite material is denoted by λ . Using the Mori–Tanaka homogenization model, the effective thermal conductivity of a composite with spherical inclusions can be expressed as following formulate.

$$\lambda = \lambda_m \left(\frac{\lambda_r + 2\lambda_m + 2V_r(\lambda_r - \lambda_m)}{\lambda_r + 2\lambda_m - V_r(\lambda_r - \lambda_m)} \right) \quad (51)$$

6.3. Physical Homogenization with Mori–Tanaka Model

The effective density of the composite depends directly on the densities of the constituent phases and their respective volume fractions. If the reinforcement phase has a lower density than the concrete matrix, increasing the reinforcement content will lead to a reduction in the overall density of the composite material, which may contribute to the development of lightweight eco-concrete materials.

$$\rho = V_r \rho_r + (1 - V_r) \rho_m \quad (52)$$

$$m = \rho V \quad (53)$$

When:

$$m = m_r + m_m \quad (54)$$

6.4. Exercises and Applications

Application 11:

Apply the Mori–Tanaka homogenization model to evaluate the effective mechanical properties of a concrete matrix ($E_m=20$ Gpa, $\nu_m=0.2$) containing porcelain particles as reinforcement ($E_r=60$ Gpa, $\nu_r=0.2$).

Considering reinforcement volume fractions V_r varying between 0% and 30%.

Table 13: The effective mechanical properties of concrete reinforced with porcelain particles.

V_r (%)	K (GPa)	G (GPa)	E (GPa)
11.11	8.33	20	0.2
5	11.68	8.76	21.03
10	12.28	9.21	22.11
15	12.91	9.68	23.24
20	13.58	10.19	24.44
25	14.29	10.71	25.71
30	15.03	11.27	27.06

The results show that the effective mechanical stiffness increases steadily with the porcelain content. The Young's modulus rises from 20.00 GPa for plain concrete to 27.06 GPa at $V_r=30\%$, which confirms that porcelain particles act as a stiff reinforcement phase.

Application 12:

Using the Mori–Tanaka homogenization model, determine the effective thermal and physical properties of an ordinary concrete matrix ($\lambda_m=1.65 \text{ W/m}^{\circ}\text{C}$, $\alpha_m=10 \times 10^{-6} \text{ }^{\circ}\text{C}^{-1}$) reinforced with cardboard waste ($\lambda_r=0.037 \text{ W/m}^{\circ}\text{C}$, $\alpha_r=9 \times 10^{-6} \text{ }^{\circ}\text{C}^{-1}$).

For reinforcement volume fractions V_r ranging from 0% to 30%.

Table 14: The effective thermal properties of eco concrete reinforced with carton (cardboard) waste.

V_r (%)	λ inf (W/m·°C)	λ sup (W/m·°C)	$\alpha \times 10^{-6}$ (C ⁻¹)
0	1.65	10	0
5	1.533	9.95	5
10	1.422	9.9	10
15	1.315	9.85	15
20	1.214	9.8	20
25	1.116	9.75	25
30	1.023	9.7	30

6.5. Limitations of the Mori-Tanaka Model

This chapter introduced the Mori–Tanaka homogenization model as an analytical method for predicting the effective properties of heterogeneous composite materials. The theoretical framework of the model was developed to evaluate the mechanical, thermal, and physical characteristics of a composite material including a concrete matrix reinforced by waste inclusions.

The essential relationships required to determine the effective elastic characteristics were initially presented. The bulk modulus, shear modulus, Young's modulus, and Poisson's ratio exemplify these connections. The macroscopic mechanical properties of the composite material can be predicted by examining the interactions between the matrix and the distributed inclusions within a representative volume element (RVE). The established relationships render this practicable.

Subsequently, the model was employed to ascertain the effective thermal properties, specifically thermal conductivity and the coefficient of thermal expansion. The results indicated that the incorporation of waste particles significantly influences the thermal behavior of the composite, particularly when the matrix and the reinforcing phase exhibit markedly different thermal properties.

The law of mixtures was utilized to determine the effective density of the composite material. This enabled the examination of the behavior of the homogenized material under real-world conditions. The results of these studies highlight the significance of the volume fraction and

the intrinsic qualities of the constituent phases in defining the overall characteristics of the composite.

Despite its foundation on fundamental concepts such as spherical inclusions, ideal surfaces, and a biphasic structure, the Mori–Tanaka model remains a valuable and often utilized method for homogenization. It provides a valuable instrument for estimating the effective characteristics of composite materials and examining the impact of reinforcing inclusions on them. This indicates that extensive experimental investigation is no longer needed.

The Mori–Tanaka homogenization method provides essential insights into the functionality of waste-reinforced concrete composites, facilitating the development of environmentally superior and optimally performing construction materials.

6.6. Conclusion of chapter 6

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The Mori–Tanaka homogenization method provides essential insights into the functionality of waste-reinforced concrete composites, facilitating the development of environmentally superior and optimally performing construction materials.

GENERAL CONCLUSION

This study analyzed various homogenization models often employed to assess the effective thermo-mechanical properties of composite materials, focusing specifically on environmentally sustainable concrete reinforced with trash inclusions. The study examined both traditional and advanced analytical techniques, including the Voigt, Reuss, Mooney, Eshelby, Hashin–Shtrikman, and Mori–Tanaka models. Each model offers a theoretical framework for forecasting the macroscopic behavior of heterogeneous materials, grounded in the intrinsic properties and volume fractions of their constituent phases.

The Voigt and Reuss models are the most basic methodologies for achieving material uniformity. They accomplish this by presuming that strain and tension are distributed equally. This offers upper and lower limits on the effective qualities of composite materials. These assumptions facilitate the research; nevertheless, they also illustrate the limitations of these models in addressing complex heterogeneous systems. They are nonetheless valuable for formulating first estimates and establishing theoretical boundaries on the probable range of actual material qualities.

The Mooney and Eshelby models exemplify complex formulations that aim to include further information regarding microstructure, including particle interactions and inclusion generation. These models offer an improved understanding of composite behavior; but, they lack practicality as they rely on simplistic assumptions on the shape,

distribution, and interaction effects of the inclusions. This is especially applicable when there is a substantial volume of reinforcement.

The Hashin–Shtrikman model imposes stringent theoretical constraints on the advantageous characteristics of biphasic materials and is regarded as one of the most dependable frameworks in micromechanics. It delineates a more precise spectrum for the prospective effective characteristics of heterogeneous composites by enforcing more stringent limitations relative to the Voigt and Reuss methodologies. The Mori–Tanaka model offers an effective analytical framework for evaluating the characteristics of composites with scattered inclusions by examining the interaction between the matrix and the reinforcing phase inside a representative volume element.

This work examines various homogenization models that demonstrate the importance of microstructural factors, such as inclusion shape, volume fraction, and matrix–inclusion interactions, in influencing the overall behavior of composite materials. Although each model employs specific simplifying assumptions, they collectively furnish essential theoretical instruments for determining the mechanical and thermal properties of environmental concrete prior to conducting any experiments. These models are essential to the development, evaluation, and enhancement of sustainable composite materials incorporating recycled or waste-derived reinforcements.

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